# Dis1

Wednesday, January 20, 2021 8:52 PM

**Agenda**
Introductions
Logistics
Problem solving

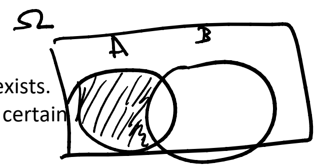
Feedback: kevin-thankyou-lin.github.io (also on staff page)
Please turn your video camera when section starts, if possible!
Breakout rooms - message me if you don't want to go into one
I'll post these as pdfs on my website

**How to prove the existence of a mathematical object (Q3)**

1. Constructive proof
… by creating that object or by giving a method to create that object

2. Non-constructive proof (e.g. Probabilistic Method)
… **without** creating that object or giving a method to create that object
e.g. Probabilistic Method

**Probabilistic Method**
If an object exists with probability greater than zero, then that object exists.
More formally, if a randomly chosen object from a sample space has a certain property with probability > 0, the object with the property exists.




1. **Miscellaneous Review**

   (a) Show that the probability that exactly one of the events $A$ and $B$ occurs is $\Pr(A) + \Pr(B) - 2\Pr(A \cap B)$.

   (b) If $A$ is independent of itself, show that $\Pr(A) = 0$ or $1$.

a) $\Pr\big( \underbrace{(A \cap B^c)}_{\phi} \overset{\text{disjoint}}{\cup} \underbrace{(A^c \cap B)}_{\phi} \big) = \Pr(A \cap B^c) + \boxed{\Pr(A^c \cap B)}$

$= \Pr(A) - \Pr(A \cap B) + \Pr(B) - \Pr(A \cap B)$

$= \underline{\qquad\qquad}$ $\square$

b) $\underbrace{\Pr(A \cap B)}_{A, B} = \Pr(A)\Pr(B)$ | $\Pr(A) = \Pr^2(A)$

$\Rightarrow \Pr(A)(1 - \Pr(A)) = 0$

e.g. exp. is throw 2 fair coins

$\Omega = \{HH, HT, TH, TT\}$

$\Omega = \{$ all config of atoms in universe where 1st coin is H, 2nd coin is H, …

2. **Balls & Bins**

   Let $n \in \mathbb{Z}_{>1}$ (i.e. $n$ is an integer greater than 1). You throw $n$ balls, one after the other, into $n$ bins, so that each ball lands in one of the bins uniformly at random.

   (a) What is an appropriate sample space to model this scenario?

   (b) What is the probability that "ball $i$ falls in bin $i$, for each $i = 1, \ldots, n$".

a) $(\frac{n}{2}, \frac{n}{2}, 0, 0, 0 \ldots 0) \in \Omega$

$\{1, 2, 3, \ldots, n\}^n$   a sample $\omega \in \Omega$, ⇒ repr. as

$\{1,2,3,\dots,n\}^n$ a sample $w \in \Omega$, is repr. as $(b_1, b_2, \dots, b_n)$ when $b_i$ is the bin that ball $i$ falls into

2nd coin is H,

b) $(1, 1, 1, \dots, 1, 2, 2, \dots, 2)$

$\frac{1}{n^n}$ $\quad \frac{1}{n} \times \frac{1}{n} \times \cdots \frac{1}{n}$

H

T,

3. **Colored Sphere**

Consider a sphere that has $\frac{1}{10}$ of its surface colored blue, and the rest is colored red. Show that, no matter how the colors are distributed, it is possible to inscribe a cube in the sphere with all of its vertices red.

*Hint: Carefully define some relevant events.*

$\Pr($ a cube we randomly inscribe to any sphere has all vertices red $) > 0$

$R_i :=$ event that cube's $i^{th}$ vertex is red

WTS

$\Rightarrow \Pr(R_1 \wedge R_2 \wedge \dots \wedge R_8) > 0$

$\Pr\left(\bigcup_{i=1}^{8} B_i\right) = \frac{8}{10}$

Define $B_i :=$ $i^{th}$ vertex is blue

$\Pr\left(\bigcap_{i=1}^{8} R_i\right) = 1 - \Pr\left(\bigcup_{i=1}^{8} B_i\right) \leq \sum_{i=1}^{8} \Pr(B_i)$

$\Pr(A) = 1 - \Pr(A^c)$ $\quad \geq 1 - 8\Pr(B_1)$

$= 1 - 8 \times \frac{1}{10} = \frac{2}{10} > 0$

4. **[Extra] The Countable Union Bound**

Let $A_1, A_2, \dots$ be a countable sequence of events. Prove that the union bound holds for countably many events:

$$\Pr\left(\bigcup_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{\infty} \Pr(A_i).$$